

Unitary representations of the quantum “ax+b” group

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January 31, 2001

Abstract

We find all unitary representations of the quantum “ax+b” group. It turns out that this quantum group is selfdual in the sense that all unitary representations are ‘numbered’ by elements of the same group. Moreover, we discover the formula for all unitary representations involving S.L. Woronowicz’s quantum exponential function.

keywords: C^* -algebra– crossed product, quantum group

MSC-class: 20G42 (Primary), 47L65, 81R15 (Secondary).

1 Introduction

Locally compact quantum groups are nowadays studied extensively by many scientists [3]. Although at the moment there is no commonly accepted definition of a locally compact quantum group, there are promising approaches and interesting examples have been worked out. Among the most interesting ones is the quantum “ax+b” group constructed by S.L. Woronowicz and S. Zakrzewski in [11]. According to the recent computation by A. van Daele, this group is the first known example of an interesting phenomenon foreseen by Vaes and Kustermans in [5]. In this paper we study the quantum ‘ax+b’ group from the point of view of unitary representations and duality theory.

The aim of this paper is to derive a formula for all unitary representations of the quantum ‘ax+b’ group. The C^* -algebra of all continuous functions vanishing at infinity on this quantum group is ‘generated’ (in the sense we explain below) by three unbonded operators $\log a$, b and $i\beta b$. We discuss the quantum ‘ax+b’ group on the Hilbert space level in Section 3. For the reader’s convenience in Section 3 and 4 we recall the definition and relevant information about the quantum ‘ax+b’ group and our previous results on unitary representations of some braided quantum groups related to the quantum ‘ax+b’ group [8]. In Section 4 we discuss the C^* - and W^* -crossed products connected with quantum ‘ax+b’ group and prove some propositions we will use later on. In Section 5 the formula for all unitary representations of quantum ‘ax+b’ group is proven. In our proof we follow the method used by S.L. Woronowicz in the case of quantum $E(2)$ group (see [12]). In the next Section we will fix the notation.

*Supported by KBN grant No 2 PO3A 036 18

2 Notation

Let \hat{q} and \hat{p} respectively denote the canonical coordinate and momentum operators acting on the Hilbert space $L^2(\mathbb{R})$. The domain of the operator \hat{q} is the set

$$D(\hat{q}) = \{ \psi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} x^2 |\psi(x)|^2 dx < \infty \}$$

and for any $\psi \in D(\hat{q})$ the operator \hat{q} is given by

$$(\hat{q}\psi)(t) = t\psi(t) .$$

The domain of the operator \hat{p} is the set

$$D(\hat{p}) = \{ \psi \in L^2(\mathbb{R}) : \psi' \in L^2(\mathbb{R}) \} ,$$

where distributional differentiation is understood.

The operator \hat{p} is given for any ψ from $D(\hat{p})$ by

$$(\hat{p}f)(x) = \frac{\hbar}{i} \frac{df(x)}{dx} ,$$

where distributional differentiation is understood.

We consider only concrete C^* -algebras, i.e. embedded into C^* -algebra of all bounded operators acting on Hilbert space \mathcal{H} , denoted by $B(\mathcal{H})$. The C^* -algebra of all compact operators acting on \mathcal{H} will be denoted by $CB(\mathcal{H})$. All algebras we consider are separable with the exception of multiplier algebras (see definition of multiplier algebra below).

Let A be C^* -algebra. Then $M(A)$ will denote the *multiplier algebra* of A , i.e.

$$M(A) = \{ m \in B(\mathcal{H}) : ma, am \in A \text{ for any } a \in A \} .$$

Observe that A is an ideal in $M(A)$. If A is a unital C^* -algebra, then $A = M(A)$, in general case $A \subset M(A)$. For example the multiplier algebra of $CB(\mathcal{H})$ is the algebra $B(\mathcal{H})$ and the multiplier algebra of C^* -algebra $C_\infty(\mathbb{R})$ of all continuous vanishing at infinity functions on \mathbb{R} is the algebra of all continuous bounded functions on \mathbb{R} denoted by $C_{bounded}(\mathbb{R})$. The natural topology on $M(A)$ is the strict topology, i.e. we say that a sequence $(m_n)_{n \in \mathbb{N}}$ of $m_n \in M(A)$ converges strictly to 0 if for every $a \in A$, we have $\|m_n a\| \rightarrow 0$ and $\|a m_n\| \rightarrow 0$, when $n \rightarrow +\infty$. Whenever we will consider continuous maps from or into $M(A)$, we will mean this topology.

For any C^* -algebras A and B , we will say that ϕ is a morphism and write $\phi \in \text{Mor}(A, B)$ if ϕ is a $*$ -algebra homomorphism acting from A into $M(B)$ and such that $\phi(A)B$ is dense in B . Any $\phi \in \text{Mor}(A, B)$ admits unique extension to a $*$ -algebra homomorphism acting from $M(A)$ into $M(B)$. For any $S \in M(A)$, operator $\phi(S)$ is given by

$$\phi(S)(\phi(a)b) = \phi(Ta)b ,$$

where $a \in A$ and $b \in B$.

For any closed operator T acting on \mathcal{H} we define its z -transform by

$$z_T = T(I + T^*T)^{-\frac{1}{2}} .$$

Observe that $z_T \in B(\mathcal{H})$ and $\|z_T\| \leq 1$. Moreover, one can recover T from z_T

$$T = z_T(I - z_T^* z_T)^{-\frac{1}{2}} .$$

A closed operator T acting on A is *affiliated* with a C^* -algebra A iff $z_T \in M(A)$ and $(I - z_T^* z_T)A$ is dense in A . A set of all elements affiliated with A is denoted by A^η . If A is a unital C^* -algebra, then $A^\eta = M(A) = A$, in general case

$$A \subset M(A) = \{T \eta A : \|T\| < \infty\} \subset A^\eta .$$

The set of all elements affiliated with $C_\infty(\mathbb{R})$ is the set of all continuous functions on real line $C(\mathbb{R})$, and a set of all elements affiliated with C^* -algebra $CB(\mathcal{H})$ is a set of all closed operators $\mathcal{C}(H)$. This last example shows that a product and a sum of two elements affiliated with A may not be affiliated with A , since it is well known that a sum and a product of two closed operators may not be closed. Affiliation relation in C^* -algebra theory was introduced by Baa and Julg in [1].

Observe, that if $\phi \in \text{Mor}(A, B)$, then one can extend ϕ to elements affiliated with A . Let us start with the observation, that for any $T \in M(A)$ we have

$$\phi(z_T) = z_{\phi(T)} .$$

Hence for any $T \eta A$ we have $z_T \in M(A)$. Moreover, there exists a unique closed operator S such that $\phi(z_T) = z_S$. This operator is given by

$$S = \phi(z_T) \phi(I - z_T^* z_T)^{-\frac{1}{2}} .$$

From now on we will write $S = \phi(T)$.

We recall now a nonstandard notion of generation we use in this paper. This notion was introduced in [14], where a generalisation of the theory of unital C^* -algebras generated by a finite number of generators was presented. It was proved in [4] that such C^* -algebras are isomorphic to algebras of continuous operator functions on compact operator domains (see Section 1.3 of [8] and references therein). In this approach, the algebra of all continuous vanishing at infinity functions on a compact quantum group is generated by matrix elements of fundamental representation. To use this approach to noncompact quantum groups, one has to extend the notion of a generation of a C^* -algebra to nonunital C^* -algebras and unbounded generators. According to the definition we recall below, *C^* -algebra of continuous vanishing at infinity functions on a locally compact quantum group* is generated by its fundamental representation. However in this case the fundamental representation is not unitary and the generators are unbounded operators, so they are not in the C^* -algebra A .

Assume for a while, that we are given a C^* -algebra A and operators T_1, T_2, \dots, T_N affiliated with A . We say that A is *generated* by T_1, T_2, \dots, T_N if for any Hilbert space \mathcal{H} , a nondegenerate C^* -algebra $B \subset B(\mathcal{H})$ and any $\pi \in \text{Mor}(A, CB(\mathcal{H}))$ we have

$$\left(\begin{array}{l} \pi(T_i) \text{ is affiliated with } A \\ \text{for any } i = 1, \dots, N \end{array} \right) \implies (\pi \in \text{Mor}(A, B)) .$$

We stress that described above 'generation' is a relation between A and some operators T_1, T_2, \dots, T_N and both have to be known in advance. There is no procedure to obtain A knowing only T_1, T_2, \dots, T_N and it is even possible that there is no A generated by such operators.

For unital C^* -algebras generation in the sense introduced above is exactly the same as the classical notion of generation. More precisely, let A be a unital C^* -algebra and let $T_1, T_2, \dots, T_N \in A$. If A is the norm closure of all linear combinations of I, T_1, \dots, T_N , then A is generated by T_1, T_2, \dots, T_N in the sense of the above definition. On the other hand, let

A be a C^* -algebra generated by $T_1, T_2, \dots, T_N \in A$, such that $\|T_i\| < \infty$ for $i = 1, 2, \dots, N$. Then A contains unity, $T_1, T_2, \dots, T_N \in A$ and A is the norm closure of the set of all linear combinations of I, T_1, T_2, \dots, T_N .

An easy example of this relation is that C^* -algebra $C_\infty(\mathbb{R})$ of all continuous vanishing at infinity functions on \mathbb{R} is generated by function $f(x) = x$ for any $x \in \mathbb{R}$. The other example is C^* -algebra $CB(L^2(\mathbb{R}))$ which is generated by \hat{p} and \hat{q} .

Let A and B be C^* -algebras and assume that we know generators of A . In order to describe $\phi \in \text{Mor}(A, B)$ uniquely it is enough to know how ϕ acts on generators of A .

We will use exclusively the minimal tensor product of C^* -algebras and it will be denoted by \otimes . We will also use the leg numbering notation. For example, if $\phi \in \text{Mor}(A \otimes A, A \otimes A)$ then $\phi_{12}(a \otimes b) = a \otimes b \otimes I_A$ and $\phi_{13}(a \otimes b) = a \otimes I_A \otimes b$ for $a, b \in A$. Clearly, $\phi_{12}, \phi_{13} \in \text{Mor}(A \otimes A, A \otimes A \otimes A)$.

Let f and ϕ be strongly commuting selfadjoint operators. Then, by the spectral theorem

$$f = \int_{\Lambda} r dE(r, \rho) \quad \text{and} \quad \phi = \int_{\Lambda} \rho dE(r, \rho) ,$$

where $dE(r, \rho)$ denotes the common spectral measure associated with f and ϕ and Λ stands for a joint spectrum of f and ϕ . Then

$$F(f, \phi) = \int_{\Lambda} F(r, \rho) dE(r, \rho) .$$

Let b be a selfadjoint operator and let the symbol χ denote the characteristic function defined on \mathbb{R} . By $\chi(b \neq 0)$ we mean the projection operator on the subspace $\ker b^\perp$, by $\chi(b < 0)$ - the projection onto the subspace on which b is negative, and so on.

3 Quantum 'ax+b' group on Hilbert space level

Now we will introduce the commutation relations describing the quantum "ax+b" group. One may guess that only two selfadjoint operators a and b will appear in such relations. However, it was found in [10] and [11], that in order to ensure existence of a selfadjoint extension of a sum $a + b$ one has to add an additional generator, denoted here by $ib\beta$.

First we will focus on the operators b and β . Operator β is an analogue of a noncontinuous (but measurable) function on the quantum "ax+b" group, so its not a good choice for generator, which should be an analogue of a continuous function. (For the notion of generator used here see Section 2.) Namely, the operator β itself will not make a good generator, since it is not affiliated with the C^* -algebra of all continuous functions vanishing at infinity on the quantum "ax+b" group, which we denote by A . Operator $ib\beta$ is affiliated with this algebra, but β is only in the von Neumann algebra A'' . That is the reason why we use $ib\beta$ instead of β . We will use the *Zakrzewski relation* introduced by S.L. Woronowicz in [10]. We say that two selfadjoint operators b and d satisfy the Zakrzewski commutation relation and we write

$$b - \circ d ,$$

if b commutes with sign d and

$$|b|^{it} d |b|^{-it} = e^{\hbar t} d \quad \text{for} \quad t \in \mathbb{R} , \quad (1)$$

where $-\pi < \hbar < \pi$. Here sign b denotes the selfadjoint, bounded operator appearing in the polar decomposition of b . Any pair of operators b and d such that $b - \circ d$ and

$\ker b = \ker d = \{0\}$ is unitarily equivalent to a direct sum of a certain number of copies of $\pm e^{\hat{q}}$ and $\pm e^{\hat{p}}$.

Consider a pair of selfadjoint operators (b, β) satisfying

$$b\beta = -\beta b \quad \beta^2 = \chi(b \neq 0) , \quad (2)$$

where the second condition means that β^2 is a projection operator onto the subspace $\ker b^\perp$ (see Section 2). If a pair of selfadjoint operators (b, β) acting on \mathcal{H} satisfies (2) we will write

$$(b, \beta) \in M_{\mathcal{H}}$$

and call (b, β) an *M-pair*.

M-pairs were studied in detail in [8]. If two pairs $(b, \beta) \in M_{\mathcal{H}}$ and $(d, \delta) \in M_{\mathcal{H}}$ satisfy an additional condition

$$b \dashv\!\circ d \quad b\delta = \delta b \quad d\beta = \beta d \quad \beta\delta = \delta\beta , \quad (3)$$

we write

$$(b, \beta, d, \delta) \in M_{\mathcal{H}}^2 .$$

For $(b, \beta, d, \delta) \in M_{\mathcal{H}}^2$ we will define the \oplus product of *M*-pairs

$$(b, \beta) \oplus (d, \delta) = (\tilde{d}, \tilde{\delta}) . \quad (4)$$

Before we write down formulas for $(\tilde{d}, \tilde{\delta})$, we will give some explanation why our choice is natural. First of all, we want $(\tilde{d}, \tilde{\delta})$ to be an *M*-pair acting on \mathcal{H} . Secondly, having already in mind application to the quantum 'ax+b' group, we define this product in such a way that the first element is as close to $b + d$ as possible. The reason why this is so desirable will be obvious in Section 5.

Here, however, we encounter a serious difficulty. It is well known that the sum of two selfadjoint operators need not to be selfadjoint, so in general the simplest proposal

$$\tilde{d} = b + d$$

will not do. The second guess is to take a selfadjoint extension of the sum above. However, in some cases, namely when deficiency indices of $b + d$ are not equal, there are no selfadjoint extensions. An easy example is to take $b = e^{\hat{q}}$ and $d = -e^{\hat{p}}$. Then $b + d$ is a closed symmetric operator, but there is no selfadjoint extension since deficiency indices are 0 and 1.

The theory of selfadjoint extensions of sums of operators satisfying Zakrzewski relation was developed by S.L. Woronowicz in [10]. It was proved there that every selfadjoint extension of $b + d$ defines uniquely a selfadjoint operator ϕ such that ϕ anticommutes with b and δ and such that $\phi^2 = \chi(e^{\frac{i\hbar}{2}} bd < 0)$.

Let F_{\hbar} denote the quantum exponential function introduced by S.L. Woronowicz in [10] as

$$F_{\hbar}(r, \rho) = \begin{cases} V_{\theta}(\log r) & \text{dla } r > 0 \text{ i } \rho = 0 \\ \{1 + i\rho|r|^{\frac{\pi}{\hbar}}\} V_{\theta}(\log|r| - \pi i) & \text{dla } r < 0 \text{ i } \rho = \pm 1 \end{cases} , \quad (5)$$

where $\theta = \frac{2\pi}{\hbar}$ and V_{θ} is a meromorphic function on \mathbb{C} defined as

$$V_{\theta}(x) = \exp \left\{ \frac{1}{2\pi i} \int_0^\infty \log(1 + a^{-\theta}) \frac{da}{a + e^{-x}} \right\} .$$

From now on we assume that

$$\hbar = \pm \frac{\pi}{2k+3}, \quad \text{where } k = 0, 1, 2, \dots \quad (6)$$

The reason why we restrict ourselves only to such values of parameter \hbar is that we are motivated by the quantum 'ax+b' group and it turns out that only for such \hbar the quantum 'ax+b' group exists on C^* -algebra and Hilbert space level (see [11]). Without loss of generality we may assume that $\ker d = \{0\}$, since in case $d = 0$ obviously there are no problems with selfadjointness of $b + d = b$. If d is invertible then the operator $e^{\frac{i\hbar}{2}} d^{-1} b$ is selfadjoint [10]. In such a case a selfadjoint extension of $b + d$ is given by

$$\tilde{d} = [b + d]_\phi = F_\hbar(f, \phi)^* d F_\hbar(f, \phi), \quad (7)$$

where

$$f = e^{\frac{i\hbar}{2}} d^{-1} b \quad \text{and} \quad \phi = (-1)^k \beta \delta \chi(e^{\frac{i\hbar}{2}} b d < 0)$$

and the relation between \hbar and k is given by (6). Analogously

$$\tilde{\delta} = F_\hbar(f, \phi)^* \delta F_\hbar(f, \phi).$$

Using Woronowicz's results we proved in [8] that if $(b, \beta, d, \delta) \in M_{\mathcal{H}}^2$ then a selfadjoint extension of $b + d$ always exists and may be given by (7). Moreover, then $([b + d]_\phi, \tilde{\delta}) \in M_H$. We also proved in that paper (Theoreme 3.3), that

Lemma 3.1 *U is a unitary representation of M acting on a Hilbert space \mathcal{K} , i.e. U is an operator map (see Section 1.3 of [8] and references therein)*

$$U : M_{\mathcal{H}} \longrightarrow B(\mathcal{K} \otimes \mathcal{H})$$

satisfying

$$U(b, \beta) U(d, \delta) = U((b, \beta) \oplus (d, \delta)),$$

iff there is $(g, \gamma) \in M_{\mathcal{K}}$ such that for any $(b, \beta) \in M_{\mathcal{H}}$

$$U(b, \beta) = F_\hbar(g \otimes b, (\gamma \otimes \beta) \chi(g \otimes b < 0)). \quad (8)$$

It means, that unitary representations of M acting on \mathcal{K} are 'numbered' by pairs $(g, \gamma) \in M_{\mathcal{K}}$. This Lemma will be of great use in the proof of Theorem 16, which is the main result of this paper.

Now we are ready to introduce commutation relations related to the quantum 'ax+b' group. We say that $(a, b, \beta) \in G_{\mathcal{H}}$ and we call (a, b, β) a G -triple if a, b and β are selfadjoint operators acting on a Hilbert space \mathcal{H} , a is positive and invertible, $a \dashv\!\!\circ b$, a commutes with β and $(b, \beta) \in M_{\mathcal{H}}$. One can define operation \oplus on G -triples for any $(a, b, \beta) \in G_{\mathcal{H}}$ and $(c, d, \delta) \in G_{\mathcal{K}}$

$$(a, b, \beta) \oplus (c, d, \delta) = (\tilde{a}, \tilde{b}, \tilde{\beta}) \in G_{\mathcal{H} \otimes \mathcal{K}}$$

by setting $\tilde{a} = a \otimes d$ and letting \tilde{b} be a selfadjoint extension of $a \otimes d + b \otimes I$ and with $\tilde{\beta}$ given by certain rather complicated formula (see [11]). Observe, that $(a \otimes d) \dashv\!\!\circ (b \otimes I)$ and that $(a \otimes d, I \otimes \delta, b \otimes I, \beta \otimes I) \in M_{\mathcal{H} \otimes \mathcal{K}}^2$. It turns out that in order to make the operation \oplus associative one has to assume that \hbar is given by (6) and that k describing selfadjoint extension of $a \otimes d + b \otimes I$ (see formula (7) and the explanation below) is related to \hbar via (6).

4 Quantum 'ax+b' group as a C^* -crossed product

We recall that A denotes the C^* -algebra of all continuous vanishing at infinity functions on quantum 'ax+b' group. The C^* -algebra A is generated (see explanation in Section 2) by unbounded operators $\log a$, b and $i\beta b$, such that $(a, b, \beta) \in G_{\mathcal{H}}$.

Assume that $\ker b = \{0\}$. This assumption is not very restrictive since every b is a direct sum of b_1 invertible and $b_2 = 0$, and the case $b_2 = 0$ is not interesting. It was proved in [11] that in that case the multiplicative unitary operator $W \in B(\mathcal{H} \otimes \mathcal{H})$ related to the quantum 'ax+b' group is given by

$$W = F_{\hbar} \left(e^{\frac{i\hbar}{2} b^{-1} a \otimes b, (-1)^k (\beta \otimes \beta) \chi(b \otimes b < 0)} \right)^* e^{\frac{i}{\hbar} \log(|b|^{-1}) \otimes \log a}. \quad (9)$$

In fact, to have a manageable multiplicative unitary, which is essential in S.L. Woronowicz theory of multiplicative unitary operators (see [15]), one needs a slightly more complicated formula for W . However, for our purpose, the formula above is good enough.

From the theory of multiplicative unitaries we know that the structure of the quantum group is encoded in its multiplicative unitary operator. More precisely, for any $d \in A$ comultiplication $\Delta \in \text{Mor}(A, A \otimes A)$ is given by

$$\Delta(d) = W(d \otimes \text{id})W^*.$$

Moreover, Δ may be extended to unbounded operators affiliated with A and is given on generators of A by

$$\Delta(a) = W(a \otimes \text{id})W^* = a \otimes a, \quad (10)$$

$$\Delta(b) = W(b \otimes \text{id})W^* = [a \otimes b + b \otimes I]_{(-1)^k (\beta \otimes \beta) \chi(b \otimes b < 0)}, \quad (11)$$

$$\Delta(ib\beta) = W(ib\beta \otimes \text{id})W^* = i\Delta(b)W(\beta \otimes \text{id})W^*.$$

By abuse of notation we also write

$$\Delta(\beta) = W(\beta \otimes \text{id})W^*. \quad (12)$$

Thus defined Δ is associative

We construct now a C^* -dynamical system. Let $\tau \in [0, +\infty[$ and

$$b(\tau) = \begin{bmatrix} \tau & 0 \\ 0 & -\tau \end{bmatrix} \quad \text{and} \quad \beta(\tau) = \begin{bmatrix} 0 & \chi(\tau \neq 0) \\ \chi(\tau \neq 0) & 0 \end{bmatrix}.$$

Then $(b, \beta) \in M_{\mathbb{C}^2}$.

Let $M_{2 \times 2}(\mathbb{C})$ denote the set of all 2×2 matrices over \mathbb{C} . Let B_o be an algebra of all continuous functions

$$f : [0, +\infty[\rightarrow M_{2 \times 2}(\mathbb{C}),$$

such that $\lim_{t \rightarrow \infty} f(t) = 0$ and $f(0)$ is a multiple of unity, i.e.

$$B_o = \{f \in C_{\infty}([0, +\infty[) \otimes M_{2 \times 2}(\mathbb{C}) \mid f(0) = zI_{M_{2 \times 2}} \quad \text{where} \quad z \in \mathbb{C} \}.$$

Any function $g \in B_o$ is given by

$$(g(b, \beta))(\tau) = (g_1(b) + g_2(b)i\beta)(\tau) = \begin{bmatrix} g_1(\tau) & ig_2(\tau) \\ -ig_2(-\tau) & g_1(-\tau) \end{bmatrix},$$

where $g_1, g_2 \in C_\infty(\mathbb{R})$, and $g_2(0) = 0$. Then B_o is a C^* -algebra. Observe that b and $i\beta b$ are affiliated with B_o and B_o is generated by b and $i\beta b$ in the sense explained in Section 2. One may identify B_o with the C^* -algebra of all continuous vanishing at infinity functions on $M_{\mathbb{C}^2}$ (see Section 1.3 of [8] and references therein). Moreover

$$M(B_o) = \{f \in C_{\text{bounded}}([0, +\infty[, M_{2 \times 2}(\mathbb{C})) \mid f(0) = zI_{M_{2 \times 2}}, \quad \text{where } z \in \mathbb{C} \}.$$

Analogously, one may identify $M(B_o)$ with the C^* -algebra of all continuous bounded functions on $M_{\mathbb{C}^2}$.

Let us introduce an action $\sigma \in \text{Aut}(M(B_o))$ of \mathbb{R} given for any function $f \in B_o$ by

$$(\sigma_t f)(\tau) = f(e^{\hbar t} \tau) \quad \text{where } \tau \in \mathbb{R}_+ \text{ and } t \in \mathbb{R}.$$

Then $(B_o, \mathbb{R}, \sigma)$ is a C^* -dynamical system (see e.g. [6]). Let us denote the C^* -crossed product algebra coming with this system by A_{cp}

$$A_{cp} = B_o \times_\sigma \mathbb{R}.$$

From the definition of C^* -crossed product follows that $M(A_{cp})$ contains a one parameter, strictly continuous group of unitary operators implementing the action σ of the group \mathbb{R} on algebra B_o . Denote the infinitesimal generator of this group by $\log a$. Then a is a strictly positive operator affiliated with A . For any $t \in \mathbb{R}$, we have that $a^{it} \in M(A)$ is a unitary operator and map $\mathbb{R} \ni t \rightarrow a^{it} d$ is continuous for any $d \in A$. The action σ of \mathbb{R} is given for any $f \in B_o$ and $t \in \mathbb{R}$ by

$$\sigma_t(f) = a^{it} f a^{-it}.$$

It follows that $a \dashv\!\circ b$ and $a\beta = \beta a$. Therefore $(a, b, \beta) \in G_{\mathcal{H}}$.

It is well-known that a linear envelope of the set

$$\{fg(\log a) : f \in B_o, g \in C_\infty(\mathbb{R})\}$$

is dense in $A = B_o \times_\sigma \mathbb{R}$. Hence $B_o \subset M(A)$. It turns out ([11, Proposition 3.1]) that $\log a$, b , and β generate A_{cp} . Moreover, Proposition 3.2 [11] says that for any triple $(\tilde{a}, \tilde{b}, \tilde{\beta}) \in G_{\mathcal{H}}$ there is a unique representation $\pi \in \text{Rep}(A_{cp}, H)$ such that $\tilde{a} = \pi(a)$, $\tilde{b} = \pi(b)$ and $\tilde{\beta} = \pi(\beta)$. It was proven in [11] that $A_{cp} = A$, i.e. that the crossed product algebra A_{cp} is the same as algebra A of all continuous, tending to 0 in infinity functions on the quantum 'ax+b' group (see [11]). From now on we will use letter A for both these algebras.

Now we will construct a dual action of \mathbb{R} on the crossed product algebra related to the dynamical system $(B_o, \mathbb{R}, \sigma)$. To this end let us consider the map

$$\theta_t = (\text{id} \otimes \varphi_t) \Delta, \tag{13}$$

where the map $\varphi \in \text{Mor}(A, C_\infty(\mathbb{R}))$ is such that for any $t \in \mathbb{R}$

$$\varphi_t(\log a) = \hbar t \quad \text{and} \quad \varphi_t(b) = 0. \tag{14}$$

From Proposition 3.2 of [11] it follows that there is only one such map. Observe that θ_t is an automorphism of A and that

$$\theta_0 = \text{id} \quad \text{and} \quad \theta_s \theta_t = \theta_{s+t}$$

The map $t \rightarrow \theta_t(d)$ is continuous for any $d \in A$. Moreover, for any $x, t \in \mathbb{R}$

$$\theta_x(a^{it}) = e^{i\hbar t x} a^{it}.$$

Thus we showed that (A, \mathbb{R}, θ) is a C^* -dynamical system and that it is the system dual to the C^* -dynamical system $(B_o, \mathbb{R}, \sigma)$.

By (10) and (14) we get

$$(\text{id} \otimes \varphi_t)\Delta(a) = (\text{id} \otimes \varphi_t)(a \otimes a) = e^{\hbar t} a . \quad (15)$$

Moreover, by (9) and (14)

$$(\text{id} \otimes \varphi_t)F_{\hbar} \left(e^{\frac{i\hbar}{2}} b^{-1} a \otimes b, (-1)^k (\beta \otimes \beta) \chi(b \otimes b < 0) \right)^* e^{\frac{i}{\hbar} \log(|b|^{-1}) \otimes \log a} = F_{\hbar}(0, 0)^* e^{it \log(|b|^{-1})} .$$

Hence by (11)

$$(\text{id} \otimes \varphi_t)\Delta(b) = b .$$

Analogously

$$(\text{id} \otimes \varphi_t)\Delta(i\beta b) = i\beta b .$$

It follows that for any $t \in \mathbb{R}$ and $g \in M(B_o)$ we have

$$\theta_t(g) = g .$$

Since $(B_o, \mathbb{R}, \sigma)$ and (A, \mathbb{R}, θ) are dual C^* -dynamical systems, it follows that $(B_o'', \mathbb{R}, \sigma)$ and $(A'', \mathbb{R}, \theta)$ are dual W^* -dynamical systems, if we only extend σ and θ appropriately.

Let \mathcal{K} be a Hilbert space, finite- or infinite-dimensional. Observe that $(B(\mathcal{K}) \otimes B_o'', \mathbb{R}, I \otimes \sigma)$ is also a W^* -dynamical system and its crossed product von Neumann algebra W^* is $B(\mathcal{K}) \otimes A''$. Analogously, the W^* -dynamical system dual to $(B(\mathcal{K}) \otimes B_o'', \mathbb{R}, I \otimes \sigma)$ is $(B(\mathcal{K}) \otimes A'', \mathbb{R}, I \otimes \theta)$.

In what follows we will need the Proposition 4.1 below, which is an easy consequence of Theorem 7.10.4 from [6]:

Proposition 4.1 *Let $m \in B(\mathcal{K}) \otimes A''$ and let for any $t \in \mathbb{R}$*

$$(\text{id} \otimes \theta_t)(m) = m .$$

Then

$$m \in B(\mathcal{K}) \otimes B_o'' .$$

We also need

Proposition 4.2 *Let B be a C^* -algebra and let $w \in M(B \otimes A)$. Then the map $\mathbb{R} \ni t \rightarrow (\text{id} \otimes \varphi_t)w \in M(B)$ is strictly continuous.*

Proof: We know that $\varphi \in \text{Mor}(A, C_{\infty}(\mathbb{R}))$ and therefore $(\text{id} \otimes \varphi)w \in M(B \otimes C_{\infty}(\mathbb{R}))$. S.L. Woronowicz in [14] showed that elements of $M(B \otimes C_{\infty}(\mathbb{R}))$ are bounded, strictly continuous functions on \mathbb{R} with values in $M(B)$. \square

5 Representation theorem

Definition 5.1 (Unitary representation) *A unitary operator $V \in M(CB(\mathcal{K} \otimes A))$ is called a (strongly continuous) unitary representation of the quantum 'ax+b' group if*

$$W_{23}V_{12} = V_{12}V_{13}W_{23} ,$$

or equivalently

$$(\text{id} \otimes \Delta)V = V_{12}V_{13} . \quad (16)$$

Observe that in case of the classical group condition (16) is equivalent to the classical definition of a unitary representation, i.e. a representation U is a map

$$U : G \ni g \rightarrow U_g \in B(\mathcal{K})$$

such that U_g is unitary for any $g \in G$, and for any $g, h \in G$ we have $U_g U_h = U_{gh}$.

It was proven in [11] that

Proposition 5.2 *Let $(a, b, \beta) \in G_{\mathcal{H}}$ and $(c, d, \delta) \in G_{\mathcal{K}}$ and let $\ker b = \{0\}$. Then the operator $V \in M(CB(\mathcal{H} \otimes A))$ given by*

$$V(\log a, b, \beta) = F_{\hbar}(d \otimes b, (\delta \otimes \beta)\chi(d \otimes b < 0))^* e^{\frac{i}{\hbar} \log c \otimes \log a} \quad (17)$$

satisfies

$$(\text{id} \otimes \Delta)V = V_{12}V_{13} ,$$

so V is a unitary representation of the quantum 'ax+b' group .

We will prove that all unitary representations of the quantum 'ax+b' group are of the form described above. This is the main result of this paper.

Theorem 5.3 *A $V \in M(CB(\mathcal{H} \otimes A))$ is a unitary representation of the quantum 'ax+b' group on a Hilbert space \mathcal{K} iff there exists $(c, d, \delta) \in G_{\mathcal{K}}$ such that*

$$V(\log a, b, \beta) = F_{\hbar}(d \otimes b, (\delta \otimes \beta)\chi(d \otimes b < 0))^* e^{\frac{i}{\hbar} \log c \otimes \log a} ,$$

where $(a, b, \beta) \in G_{\mathcal{H}}$ and $\log a, b$ and $ib\beta$ are the generators of A .

Proof: Let V be a unitary representation of the quantum 'ax+b' group on a Hilbert space \mathcal{K} . Then for any $t \in \mathbb{R}$ the operator $(\text{id} \otimes \varphi_t)V \in B(\mathcal{K})$ is unitary.

Applying $(\text{id} \otimes \varphi_s \otimes \varphi_t)$ to both sides of (16) we get

$$V(\hbar(s+t), 0, 0) = V(\hbar s, 0, 0)V(\hbar t, 0, 0)$$

Hence

$$(\text{id} \otimes \varphi_{s+t})V = (\text{id} \otimes \varphi_s)V(\text{id} \otimes \varphi_t)V$$

i.e. $(\text{id} \otimes \varphi)V$ is a representation of \mathbb{R} .

The strict topology coincides on $B(H) = M(CB(H))$ with the *-strong operator topology. Since *-strong operator topology is stronger than strong operator topology, Proposition 4.2 shows that the map

$$\mathbb{R} \ni t \rightarrow (\text{id} \otimes \varphi_t)V \in B(\mathcal{K})$$

is strongly continuous. Therefore, by Stone's Theorem, there is a selfadjoint, strictly positive operator c acting on \mathcal{K} and such that

$$(\text{id} \otimes \varphi_t)V = e^{it \log c} \quad (18)$$

for any $t \in \mathbb{R}$.

Observe that by (13), (16) and (18)

$$(\text{id} \otimes \theta_t)V = V(e^{it \log c} \otimes \text{id}) .$$

Moreover, by (15)

$$(\text{id} \otimes \theta_t) e^{-\frac{i}{\hbar} \log c \otimes \log a} = (e^{-it \log c} \otimes \text{id}) e^{-\frac{i}{\hbar} \log c \otimes \log a}$$

Hence

$$(\text{id} \otimes \theta_t) V e^{-\frac{i}{\hbar} \log c \otimes \log a} = V e^{-\frac{i}{\hbar} \log c \otimes \log a} .$$

Observe that

$$V e^{-\frac{i}{\hbar} \log c \otimes \log a} \in B(\mathcal{K}) \otimes A'' .$$

By Proposition 4.1

$$V e^{-\frac{i}{\hbar} \log c \otimes \log a} = f(b, \beta) ,$$

where $f \in B(\mathcal{K}) \otimes B''_o$. It means that one can ([14]) identify f with a measurable (operator) function $f : M \rightarrow B(\mathcal{K} \otimes \mathcal{H})$. Because V is unitary and

$$V = f(b, \beta) e^{\frac{i}{\hbar} \log c \otimes \log a} ,$$

it follows that f may be identified with a continuous and bounded (operator) function with values in unitary operators acting on \mathcal{K} . Let us compute

$$\begin{aligned} (\text{id} \otimes \Delta) V &= (\text{id} \otimes \Delta) \left(f(b, \beta) e^{\frac{i}{\hbar} \log c \otimes \log a} \right) = f(\Delta(b), \Delta(\beta)) e^{\frac{i}{\hbar} \log c \otimes \log \Delta(a)} = \\ &= f([a \otimes b + b \otimes I]_\tau, \Delta(\beta)) e^{\frac{i}{\hbar} \log c \otimes \log(a \otimes a)} . \end{aligned} \quad (19)$$

Applying $(\text{id} \otimes \varphi_t \otimes \text{id})$ to both sides of (16) we get

$$(\text{id} \otimes \varphi_t \otimes \text{id}) V_{12} V_{13} = (e^{it \log c} \otimes \text{id}) V$$

and

$$(\text{id} \otimes \varphi_t \otimes \text{id})(\text{id} \otimes \Delta) V = f(e^{\hbar t} b, \beta) (e^{\frac{i}{\hbar} \log c \otimes \log a}) (e^{it \log c} \otimes \text{id}) .$$

Comparing the resulting expressions we obtain

$$(e^{it \log c} \otimes \text{id}) V = f(e^{\hbar t} b, \beta) (e^{\frac{i}{\hbar} \log c \otimes \log a}) (e^{it \log c} \otimes \text{id}) . \quad (20)$$

On the other hand, applying $(\text{id} \otimes \text{id} \otimes \varphi_t)$ to both sides of (16) we get

$$(\text{id} \otimes \text{id} \otimes \varphi_t) V_{12} V_{13} = V (e^{it \log c} \otimes \text{id})$$

and

$$(\text{id} \otimes \text{id} \otimes \varphi_t)(\text{id} \otimes \Delta) V = f(b, \beta) (e^{\frac{i}{\hbar} \log c \otimes \log a}) (e^{it \log c} \otimes \text{id}) .$$

Comparing the resulting expressions we obtain

$$V (e^{it \log c} \otimes \text{id}) = f(b, \beta) (e^{\frac{i}{\hbar} \log c \otimes \log a}) (e^{it \log c} \otimes \text{id}) . \quad (21)$$

We insert now $\log a$ instead of $\hbar t$ in formulas (20) and (21) we get

$$\begin{aligned} (e^{\frac{i}{\hbar} \log c \otimes \log a} \otimes \text{id}) V_{13} &= f(a \otimes b, I \otimes \beta) (e^{\frac{i}{\hbar} \log c \otimes \log a})_{13} (e^{\frac{i}{\hbar} \log c \otimes \log a})_{12} \\ V_{12} (e^{\frac{i}{\hbar} \log c \otimes \log a})_{13} &= f(b \otimes I, \beta \otimes I) (e^{\frac{i}{\hbar} \log c \otimes \log a})_{12} (e^{\frac{i}{\hbar} \log c \otimes \log a})_{13} . \end{aligned}$$

Then

$$V_{12} = f(b \otimes I, \beta \otimes I) (e^{\frac{i}{\hbar} \log c \otimes a})_{12} \quad (22)$$

$$V_{13} = (e^{\frac{i}{\hbar} \log c \otimes a})_{12}^* f(a \otimes b, I \otimes \beta) (e^{\frac{i}{\hbar} \log c \otimes a})_{13} (e^{\frac{i}{\hbar} \log c \otimes a})_{12} . \quad (23)$$

remembering that V satisfies $(\text{id} \otimes \Delta)V = V_{12}V_{13}$ and using (19), (22) and (23), we get

$$f([a \otimes b + b \otimes I]_\tau, \Delta(\beta)) e^{\frac{i}{\hbar} \log c \otimes \log(a \otimes a)} = f(b \otimes I, \beta \otimes I) f(a \otimes b, I \otimes \beta) (e^{\frac{i}{\hbar} \log c \otimes \log a})_{13} (e^{\frac{i}{\hbar} \log c \otimes \log a})_{12} .$$

Moreover

$$e^{\frac{i}{\hbar} \log c \otimes \log(a \otimes a)} = e^{\frac{i}{\hbar} \log c \otimes \{\log(I \otimes a) + \log(a \otimes I)\}} = (e^{\frac{i}{\hbar} \log c \otimes \log a})_{13} (e^{\frac{i}{\hbar} \log c \otimes \log a})_{12} .$$

Therefore

$$f([a \otimes b + b \otimes I]_\tau, \Delta(\beta)) = f(b \otimes I, \beta \otimes I) f(a \otimes b, I \otimes \beta)$$

or equivalently

$$f([a \otimes b + b \otimes I]_\tau, \Delta(\beta))^* = f(a \otimes b, I \otimes \beta)^* f(b \otimes I, \beta \otimes I)^* . \quad (24)$$

Using (12) and (9) and remembering that β commutes with $s|b|^{-1}$, we get

$$\Delta(\beta) = \quad (25)$$

$$= F_h \left(e^{\frac{i\hbar}{2} b^{-1} a \otimes b, (-1)^k (\beta \otimes \beta) \chi(b \otimes b < 0)} \right)^* (\beta \otimes I) F_h \left(e^{\frac{i\hbar}{2} b^{-1} a \otimes b, (-1)^k (\beta \otimes \beta) \chi(b \otimes b < 0)} \right) .$$

Let us introduce the notation

$$R = a \otimes b , \quad \rho = I \otimes \beta \quad S = b \otimes I , \quad \sigma = \beta \otimes I . \quad (26)$$

We see that $(R, \rho, S, \sigma) \in M_{\mathcal{H} \otimes \mathcal{H}}^2$. Moreover, if we define

$$T = e^{\frac{i\hbar}{2}} S^{-1} R = e^{\frac{i\hbar}{2} b^{-1} a \otimes b} \quad \text{and} \quad \tau = (-1)^k (\beta \otimes \beta) , \quad (27)$$

then also $(T, \tau) \in M_{\mathcal{H}}$. Since $a \dashv b$, then by Theorem 6.1 from [10]

$$\text{sign} (e^{\frac{i\hbar}{2} b^{-1} a}) = \text{sign} (b) \text{sign} (a) .$$

Since $a > 0$, we finally get

$$\chi(T < 0) = \chi(e^{\frac{i\hbar}{2} b^{-1} a \otimes b} < 0) = \chi(b \otimes b < 0) .$$

Inserting (26) and (27) into formula (25) and using (5) we get

$$\Delta(\beta) = F_h(T, \tau \chi(T < 0))^* \sigma F_h(T, \tau \chi(T < 0)) .$$

Moreover, we observe that the following selfadjoint extensions are the same

$$[a \otimes b + b \otimes I]_\tau = [R + S]_{\tau \chi(T < 0)} .$$

Using two last formulas and substituting (26) into the right hand side of (24), one can write (24)

$$f((R, \rho) \oplus_M (S, \sigma))^* = f(R, \rho)^* f(S, \sigma)^* .$$

From Theorem 3.3 [8] (recalled in Section 2) follows that if a function f is a measurable (operator) function $f : M \rightarrow B(\mathcal{K} \otimes \mathcal{H})$ and satisfies the above condition, then it is given by

$$f(b, \beta) = F_h(d \otimes b, (\delta \otimes \beta) \chi(d \otimes b < 0))^* ,$$

where $(d, \delta) \in M_K$.

Therefore

$$V = F_{\hbar}(d \otimes b, (\delta \otimes \beta)\chi(d \otimes b < 0))^* e^{\frac{i}{\hbar} \log c \otimes \log a} \quad (28)$$

What is left to prove is that

$$c - \circ d \quad \text{and} \quad \delta c = c \delta .$$

To this end, let us observe that combining (28) and (23) we get

$$\begin{aligned} & F_{\hbar}(d \otimes I \otimes b, (\delta \otimes I \otimes \beta)\chi(d \otimes I \otimes b < 0))^* = \\ & = (e^{\frac{i}{\hbar} \log c \otimes \log a})_{12}^* F_{\hbar}(d \otimes a \otimes b, (\delta \otimes I \otimes \beta)\chi(d \otimes I \otimes b < 0))^* (e^{\frac{i}{\hbar} \log c \otimes \log a})_{12} . \end{aligned}$$

Hence

$$(d \otimes \text{id}) = e^{-\frac{i}{\hbar} \log c \otimes \log a} (d \otimes a) e^{\frac{i}{\hbar} \log c \otimes \log a} \quad (29)$$

and

$$(\delta \otimes \text{id}) = e^{-\frac{i}{\hbar} \log c \otimes \log a} (\delta \otimes I) e^{\frac{i}{\hbar} \log c \otimes \log a} . \quad (30)$$

Substituting $a = e^{\hbar k} I$ in formula (29) we get $c - \circ d$. Moreover, after the same substitution in formula (30) we see that c commutes with δ . Hence $(c, d, \delta) \in G_K$, which completes the proof. \square

Acknowledgments

This is a part of the author's Ph.D. thesis [9] written under the supervision of Professor Stanisław L. Woronowicz at the Department of mathematical Methods in Physics at the Warsaw University. The author is greatly indebted to Professor Stanisław L. Woronowicz for stimulating discussions and important hints and comments. The author also wishes to thank Professor Wiesław Pusz and Marek Bożejko for several helpful suggestions and Piotr Sołtan for reading carefully the manuscript.

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